

ON AN APPROACH TO CONSTRUCTING STATIC BALL MODELS IN GENERAL RELATIVITY ¹

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Abstract – An approach to construction of static models is demonstrated for a fluid ball. Five examples are considered. Two of them are exact solutions of the Einstein equations; the other three are connected with the Airy special functions, the hypergeometric functions and the Heun functions.

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1. Introduction

One of the most important problems in general relativity is that of finding exact solutions of the gravitational equations. Unfortunately, this task is not easy. Another way is an analytic construction of solutions to the gravitational equations with a certain physical interpretation.

Further on we take the metric in Bondi's form

$$ds^2 = G(r)^2 dt^2 + 2L(r) dt dr - r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \quad (1)$$

where G^2, L are metric functions, r is a radial variable and θ, φ are spherical angles; the speed of light and Newton's constant of gravity are put equal to unit.

The gravitational field is described by the metric tensor g_{ik} , which can be found from Einstein's equations

$$R_{ik} - \frac{1}{2}Rg_{ik} = -\kappa T_{ik}, \quad (2)$$

where $i, j, k = 0, 1, 2, 3$; R_{ik} is the Ricci tensor, R is the scalar curvature of space-time; κ is Einstein's gravitational constant.

The energy-momentum tensor (EMT) of Pascal's perfect fluid can be written as

$$T_{ik} = (p(r) + \mu(r)) \cdot u_i u_k - p(r) g_{ik}, \quad (3)$$

where $p(r)$ is the pressure, $\mu(r)$ is the mass density, u_i is the 4-velocity.

The gravitational equations in dimensionless variables can be reduced after elementary transformations to the form

$$\varepsilon(x) = 1 - \frac{\chi}{x} \cdot \int \mu(x) \cdot x^2 dx; \quad (4)$$

$$G'' + \left(\frac{\varepsilon'}{2 \cdot \varepsilon} - \frac{1}{x} \right) G' + \left(\frac{\varepsilon'}{2 \cdot x \cdot \varepsilon} + \frac{1 - \varepsilon}{x^2 \cdot \varepsilon} \right) G = 0; \quad (5)$$

$$p' = -\frac{1}{2 \cdot \varepsilon} \cdot \left(\chi x p + \frac{1 - \varepsilon}{x} \right) \cdot (\mu + p), \quad (6)$$

where $x = r/R$ is the dimensionless radius; $0 \leq x \leq 1$; differentiation in x is denoted by a prime; R is the radius of the astrophysical object; $\chi = \kappa \cdot R^2$;

$$\varepsilon(x) = \frac{G^2(x)}{L^2(x)}. \quad (7)$$

2. Reduction of the Einstein equations

Eq (5) can be reduced to an oscillatory-type equation

$$\frac{d^2 G}{d\zeta^2} + \Omega^2(\zeta(y))G = 0, \quad (8)$$

where

$$\Omega^2 = -d(\Phi/y)/dy, \quad (9)$$

$y = x^2$, ζ is a new variable:

$$d\zeta = y dy / (2\sqrt{\varepsilon}). \quad (10)$$

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Now we introduce $\varepsilon = 1 - \Phi$ as in [1], where

$$\Phi = (\chi/(2\sqrt{y})) \int \mu \sqrt{y} dy, \quad (11)$$

where $\chi = \kappa R^2$, Φ is an analog of Newton's gravitational potential.

Further we present the function Ω^2 in the form of a power series:

$$\Omega^2(y) = \sum_{n=0}^{\infty} a_n y^n. \quad (12)$$

Now we can find both Φ and μ in the general case from (8) and (10) with help of (11)

$$\Phi = (\mu_0/3)y - \sum_{n=0}^{\infty} \frac{a_n}{n+1} y^{n+2}; \quad (13)$$

$$\mu/\mu_0 = 1 - (1/\mu_0) \sum_{n=0}^{\infty} \frac{2n+3}{n+1} a_n y^{n+1}, \quad (14)$$

where μ_0 is the central mass density and the coefficients a_n are constants to be found from boundary conditions.

3. Construction of solutions to the gravitational equations

Consider the power series (11). At first we will take all coefficients a_n equal to zero. After that we will consider only $a_0 \neq 0$, further $a_0 \neq 0$, $a_1 \neq 0$ and so on. In each case we will construct the corresponding mass density and make an attempt to find the function G from (7). Here we must remark that the metric functions $g_{00} = G^2$ and $g_{10} = g_{01} \equiv L = G/\sqrt{\varepsilon}$.

1. If we take all $a_n = 0$, then $\Omega^2 = 0$ and Eq.(8) is transformed to

$$\frac{d^2 G}{d\zeta^2} = 0. \quad (15)$$

The solution is $G(\zeta) = C_1 \cdot \zeta + C_2$, where C_1, C_2 are integration constants. From (14) we have $\mu \equiv \mu_0$. In other words, it is the Schwarzschild interior ball model.

2. If we include only $a_0 \neq 0$, then $\Omega^2 \equiv \Omega_0^2 = \text{const}$ and

$$G \propto \cos(\Omega_0 \zeta + \varphi_0). \quad (16)$$

In this case

$$\mu = \mu_0 - 3a_0 y = \mu_0 - 3a_0 x^2, \quad (17)$$

i.e. we have a parabolic distribution of the mass density (see [1]).

We must say that in these cases the approximate solutions coincide with the exact well-known solutions of the Einstein equations.

3. Now we take ζ approximately as $\zeta \approx y/2$ ($\zeta(0) = 0$), because there is the difficulty in determining the

variable ζ via y . Here we have an approximate estimate because the variable is $y \ll 1$. In this case the origins of the two variables y and ζ are glued.

The further approximation will be

$$\Omega^2 = a_0 + a_1 y \approx a_0 + 2a_1 \zeta, \quad (18)$$

and Eq.(7) can be written as

$$\frac{d^2 G}{d\zeta^2} + (a_0 + 2a_1 \zeta)G = 0, \quad (19)$$

while the mass density is

$$\begin{aligned} \mu &\approx \mu_0 - 6a_0 \zeta - 10a_1 \zeta^2 \\ &\approx \mu_0 - 3a_0 x^2 - (5/2)a_1 x^4. \end{aligned} \quad (20)$$

The solution of Eq.(19) is

$$\begin{aligned} G &= C_1 \text{AiryAi}\left(\frac{a_0 + 2a_1 \zeta}{(2a_1)^{2/3}}\right) + \\ &+ C_2 \text{AiryBi}\left(\frac{a_0 + 2a_1 \zeta}{(2a_1)^{2/3}}\right), \end{aligned} \quad (21)$$

where *AiryAi* and *AiryBi* are the Airy special functions, and C_1, C_2 are constants.

4. The following real solution with

$$\Omega^2 \approx a_0 + 2a_1 \zeta - 4a_2 \zeta^2 \quad (22)$$

and

$$\begin{aligned} \mu &\approx \mu_0 - 6a_0 \zeta - 10a_1 \zeta^2 + (56/3)a_2 \zeta^3 \\ &\approx \mu_0 - 3a_0 x^2 - (5/2)a_1 x^4 + (7/3)a_2 x^6 \end{aligned} \quad (23)$$

can be written as the linear combination of hypergeometric functions:

$$\begin{aligned} G &= [C_1 \text{hypergeom}(\alpha_1, \beta_1, \gamma(\zeta)) + \\ &C_2 \text{hypergeom}(\alpha_2, \beta_2, \gamma(\zeta))(4a_2 \zeta - a_1)] \\ &\times \exp(\delta(\zeta)), \end{aligned} \quad (24)$$

where C_1, C_2 are constants,

$$\alpha_1 = -(a_1/4 + a_0 a_2 - 2a_2^{3/2})/(8a_2^{3/2});$$

$$\beta_1 = 1/2; \quad \alpha_2 = \alpha_1 + 1/2;$$

$$\beta_2 = \beta_1 + 1; \quad \gamma(\zeta) = (4a_2 \zeta - a_1)^2/(8a_2^{3/2});$$

$$\delta(\zeta) = \zeta(a_1/2 - a_2 \zeta)/\sqrt{a_2}.$$

5. One more real solution can be found for

$$\Omega^2 \approx a_0 + 2a_1 \zeta - 4a_2 \zeta^2 - 16a_4 \zeta^4 \quad (25)$$

$$\begin{aligned} \mu &\approx \mu_0 - 6a_0 \zeta - 10a_1 \zeta^2 + (56/3)a_2 \zeta^3 \\ &- (352/5)a_4 \zeta^5 \approx \mu_0 - 3a_0 x^2 - (5/2)a_1 x^4 \\ &+ (7/3)a_2 x^6 + (11/5)a_4 x^{10}. \end{aligned}$$

This solution is written as a linear combination of HeunT functions

$$G = C_1 HeunT(\alpha, -\beta, \gamma, -b \cdot \zeta) \cdot \exp(\psi(\zeta)) + \\ + C_2 HeunT(\alpha, \beta, \gamma, b \cdot \zeta) \cdot \exp(-\psi(\zeta)), \quad (26)$$

where C_1, C_2 are constants,

$$\alpha = 3^{2/3}(4a_4a_0 + a_2^2)/(16a_4^{4/3}); \quad \beta = 3a_1/(4\sqrt{a_4}),$$

$$\gamma = 3^{1/3}a_2/(2a_4^{2/3}), b = (2/3)3^{2/3}a_4^{1/6},$$

$$\psi(\zeta) = \zeta \cdot (3a_2 + 8a_4\zeta^2)/(6\sqrt{a_4}).$$

The pressure can be found from the equation Eq.(6) and the metric function L from (7).

4. Summary

In conclusion, we must note that an approach to the construction of the ball static models is demonstrated in this paper. This approach is based on the reduction of gravitational equations to the oscillatory-type equation and the using the expansion in the power series the function which plays a role of the frequency. Main difficulty is to find the new variable through the dimensionless radial variable. Five examples are considered. Two from them are the exact solutions of the Einstein equations for a fluid ball. The third, fourth and fifth examples are connected with the special Airy functions, with the hypergeometric functions and the HeunT functions.

References

- [1] A.M.Baranov, Vestnik of Krasnoyarsk State University (Phys. & Math. Sci.), No.1, 5-12 (2002)(in Russian)).